M-math 2nd year Final Exam
Subject : Fourier Analysis

Time : 3.00 hours Max.Marks 60.
Answer any two questions from each part. Contact : rajeevbee@hotmail.com

Part A

1. Let $f(t) := \int_{-\pi}^{t} f'(s)ds$ where $f'(s)$ is continuous on $[-\pi, \pi]$. Show that $\lim_{n\to\infty} S_n f(t) = f(t)$ for every $t \in (-\pi, \pi]$, where $S_n f(t) = f * D_n(t)$ and $D_n(t)$ is the Dirichlet kernel. (15)

2. With the $S_n f$ as above , and $f \in L^1$ show that

$$\lim_{n\to\infty} \int_a^b S_n f(t)dt = \int_a^b f(t)dt.$$

Hint : Consider the Fourier series of $F(t) := \int_{-\pi}^{t}(f(s) - \hat{f}(0))ds$. (15)

3. Let $f \in L^2(-\pi, \pi]$. Then show that f is absolutely continuous i.e. $f(t) = \int_{-\pi}^{t} f'(s)ds$ and $f' \in L^2$ if and only if $\sum_{-\infty}^{\infty} n^2|\hat{f}(n)|^2 < \infty$. (15)

Part B

4. Let $f := I_{[a,b]}$. Then show that the Hilbert transform Hf of f is given by $Hf(x) = \frac{1}{\pi} log \frac{|x-a|}{|x-b|}$ for $x \neq a, b$. Show also that $Hf \notin L^1(\mathbb{R})$. (9+6.)

5. Let $M > 0$ and $K_M(x) := \frac{1-cos 2M\pi x}{2M\pi^2 x^2}, x \neq 0; K_M(0) = M$. Let $f \in L^1(\mathbb{R})$ satisfy $\lim_{M\to\infty} M\|f * K_M - f\|_{L^1} = 0$. Show that $f = 0$ *a.e.* Hint : Find an estimate for $|\xi \hat{f}(\xi)|$ in terms of $\|K_M * f - f\|_{L^1}$. (15)

6. Suppose $K_t(x)$is an approximate identity on $\mathbb{R}$ and t in some index set. Suppose $f(x)$ is a bounded function on $\mathbb{R}$ with $\lim_{x\to 0} f(x) = L$. Show that $\lim_{t} \int K_t(x)f(x)dx = L$. (15)